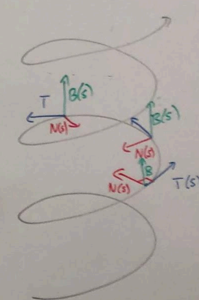
Lesson 9 Curvature, Torsion, Frenet.

Defn: A curve $\vec{x}: \mathbb{R} \to \mathbb{R}^3$ is parametrized by arclengh if $|x'(s)| = 1 \ \forall s \in \mathbb{R}$.



In this setting

$T(s) = x'(s)$ no need to divide by speed=1

$N(s) = T'(s) / |T'(s)|$ to be unit length.

$B(s) = T(s) \times N(s)$ perpendicular to both and unit length

Example:

$$C(s) = (r\cos(\ell s), r\sin(\ell s), h\cdot \ell s)$$

Lets choose $\ell = \boxed{(r^2+h^2)^{-1/2}} > 0$ ✓

so that $|C'(s)| = 1$.

$$C'(s) = (\underbrace{-r\sin(\ell s)\cdot \ell}_{\text{chain rule}}, r\cos(\ell s)\cdot \ell, h\ell)$$

$$|C'(s)| = \sqrt{(r\ell)^2\sin^2(\ell s) + (r\ell)^2\cos^2(\ell s) + (h\ell)^2}$$

$$1 = \sqrt{(r\ell)^2(1) + (h\ell)^2}$$

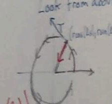
$$1 = \ell\sqrt{r^2+h^2} \quad \text{So } \ell = \frac{1}{\sqrt{r^2+h^2}}$$

$$T(s) = C'(s) = \ell \begin{pmatrix} -r\sin(\ell s) \\ r\cos(\ell s) \\ h \end{pmatrix}$$

(tangent to cylinder)

Don't divide by $|C'(s)| = 1$

Look from above



$$N(s) = \frac{T'(s)}{|T'(s)|}$$

where $T'(s) = \ell \begin{pmatrix} -r\ell\cos(\ell s) \\ -r\ell\sin(\ell s) \\ 0 \end{pmatrix}$ because h is constant

$$\text{So } |T'(s)| = |\ell|\sqrt{(-r\ell\cos(\ell s))^2 + (-r\ell\sin(\ell s))^2 + 0^2} = \ell r \ell = r\ell^2$$

$$= \ell\sqrt{(r\ell)^2(\underbrace{\cos^2(\ell s) + \sin^2(\ell s)}_{1})}$$

$$\text{So } N(s) = \frac{1}{r\ell}(-r\ell\cos(\ell s), -r\ell\sin(\ell s), 0)$$

is horizontal.

$$= (-\cos(\ell s), -\sin(\ell s), 0)$$

Example:

$$C(s) = (r\cos(\ell s), r\sin(\ell s), h \cdot \ell s)$$

Lets choose $\ell = \boxed{(r^2+h^2)^{-1/2}} > 0$ ✓

$$T(s) = C'(s) = \ell \begin{pmatrix} -r\sin(\ell s) \\ r\cos(\ell s) \\ h \end{pmatrix}$$

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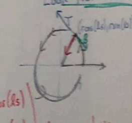


$$N(s) = \frac{T'(s)}{|T'(s)|}$$

where $T'(s) = \ell \begin{pmatrix} -r\ell\cos(\ell s) \\ -r\ell\sin(\ell s) \\ 0 \end{pmatrix}$ because h is constant

$$\text{So } |T'(s)| = |\ell| \sqrt{(-r\ell\cos(\ell s))^2 + (-r\ell\sin(\ell s))^2 + 0^2}$$

$$= \ell \sqrt{(r\ell)^2 \underbrace{(\cos^2(\ell s) + \sin^2(\ell s))}_{1}} = \ell r \ell = r\ell^2$$

$$\text{So } N(s) = \frac{1}{r\ell}(-r\ell\cos(\ell s), -r\ell\sin(\ell s), 0)$$

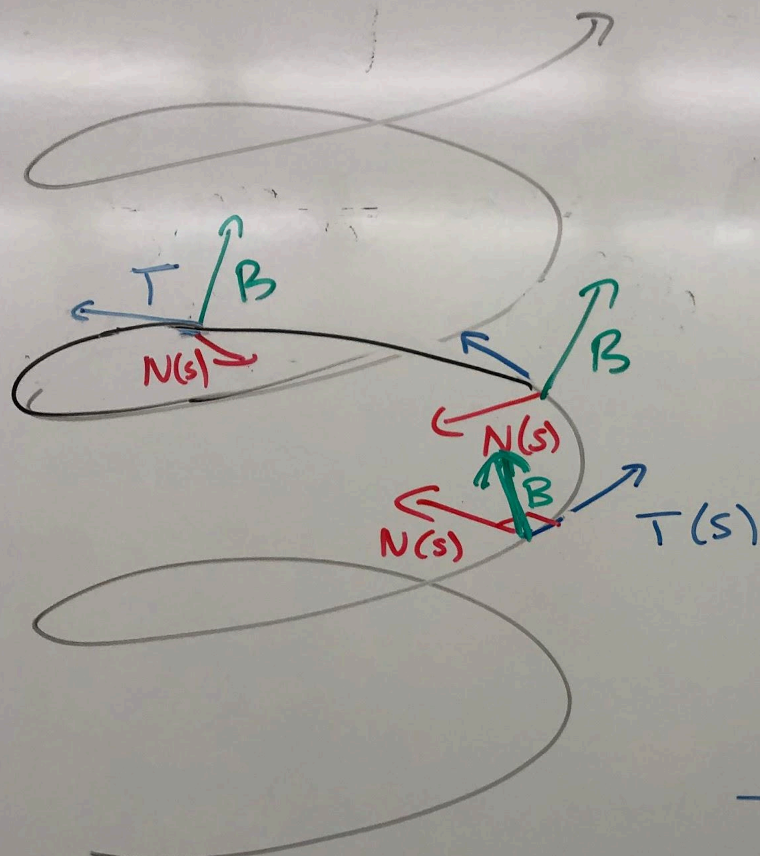
is horizontal.

$$= (-\cos(\ell s), -\sin(\ell s), 0)$$

Binormal

$$B = T \times N = \begin{pmatrix} -r\ell\sin(\ell s) \\ r\ell\cos(\ell s) \\ h\ell \end{pmatrix} \times \begin{pmatrix} -\cos(\ell s) \\ -\sin(\ell s) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{r\ell\cos(\ell s)\cdot 0} - h\ell(-\sin(\ell s)) \\ h\ell(-\cos(\ell s)) - \cancel{(-r\ell\sin(\ell s))\cdot 0} \\ -r\ell\sin(\ell s)(-\sin(\ell s)) - r\ell\cos(\ell s)(-\cos(\ell s)) \end{pmatrix} = \begin{pmatrix} h\ell\sin(\ell s) \\ -h\ell\cos(\ell s) \\ r\ell \end{pmatrix}$$



T N B

- Frenet Frame
  three orthonormal
  vectors.

Lesson 9 Curvature, Torsion, Frenet.

Defn Curvature $\kappa = \left|\frac{dT}{ds}\right|$

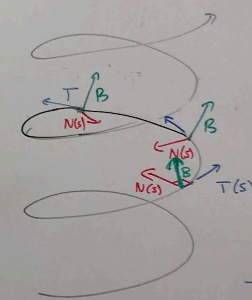
In our example

$$\kappa = r\ell^2 = \frac{r}{r^2+h^2}$$

$$\approx \frac{1}{r} \text{ for } r \text{ large and } h \text{ fixed.}$$

$$= \frac{1}{r} \text{ for any } r \text{ if } h=0$$

To define Torsion we will need Vector Calc and Linear Algebra.



Frenet Frame $\sim$ T N B

three orthonormal vectors.

Review Linear Algebra

Defn A 3×3 orthogonal matrix M has orthonormal rows.

which means $M = \begin{pmatrix} T \\ N \\ B \end{pmatrix}$

where $T\cdot N = N\cdot B = B\cdot T = 0$

and $T\cdot T = N\cdot N = B\cdot B = 1$

Thm: If M is an orthogonal matrix then M^T is its inverse

Pf: $\begin{pmatrix} T \\ N \\ B \end{pmatrix}\begin{pmatrix} T & N & B \end{pmatrix} = \begin{pmatrix} T\cdot T & T\cdot N & T\cdot B \\ N\cdot T & N\cdot N & N\cdot B \\ B\cdot T & B\cdot N & B\cdot B \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Defn The transpose A^T of a matrix A changes rows to columns so

$A_{ij} \longrightarrow A^T_{ji}$

Thm $(AB)^T = B^T A^T$

Defn: A matrix A is symmetric if $A^T = A$

$A_{ij} = A_{ji}$ So: $\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$

Defn A matrix A is skewsymmetric if $A^T = -A$

$A_{ij} = -A_{ji}$ So: $\begin{pmatrix} 0 & b & c \\ -b & 0 & d \\ -c & -d & 0 \end{pmatrix}$

So $A_{11} = -A_{11}$ implies $A_{11} = 0$

Similarly $A_{22} = A_{33} = 0$

Frenet-Serret Formula

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

where $T = C'(s)$

$N = \frac{1}{\kappa} T'(s)$

$\kappa = |T'(s)|$

$B = T \times N$

Skew Symmetric

If mult both sides by $\begin{pmatrix} T & N & B \end{pmatrix}$ we see this matrix is $\begin{pmatrix} T \\ N \\ B \end{pmatrix}' \begin{pmatrix} T & N & B \end{pmatrix}$

So last two entries must be something we call torsion τ.

So $N' = -\kappa T + \tau B$

and $B' = -\tau N$

$\tau = |B'|$

Torsion

Proof that we have the green part:

$$\boxed{T' = \kappa N = 0T + \kappa N + 0B}$$

$(T_1' \ T_2' \ T_3') =$

$= \kappa(N_1, N_2, N_3) = (0T_1 + \kappa N_1 + 0B_1, 0T_2 + \kappa N_2 + 0B_2, 0T_3 + \kappa N_3 + 0B_3)$

Proof that the matrix is skew symmetric

Start with $\begin{pmatrix} T \\ N \\ B \end{pmatrix}$ is orthogonal.

So $\begin{pmatrix} T \\ N \\ B \end{pmatrix} \begin{pmatrix} T & N & B \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Implicit Diff Both sides:

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' \begin{pmatrix} T & N & B \end{pmatrix} + \begin{pmatrix} T \\ N \\ B \end{pmatrix} \begin{pmatrix} T & N & B \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} T \\ N \\ B \end{pmatrix}' \begin{pmatrix} T & N & B \end{pmatrix}} = -\begin{pmatrix} T \\ N \\ B \end{pmatrix} \begin{pmatrix} T & N & B \end{pmatrix}'$$

$$= -\left(\begin{pmatrix} T \\ N \\ B \end{pmatrix}' \begin{pmatrix} T & N & B \end{pmatrix} \right)^T$$

So it is skew symmetric

Taylor Series about $C(0)$

$$\underbrace{\underbrace{\vec{C}(0) + s\,\vec{T}(0)}_{\substack{1^{st}\text{order}\\ \text{tan line}}} + \frac{s^2}{2}\kappa(0)\vec{N}(0)}_{\substack{2^{nd}\text{ order}\\ \text{is a parabola}\\ \text{in the } \vec{T}\text{-}\vec{N} \text{ plane}\\ \text{"osculating plane"}\\ \text{is perp to } \vec{B}}} - \frac{s^3}{6}\left(\kappa^2(0)\vec{T}(0) + \kappa'(0)\vec{N}(0) + \kappa(0)\tau(0)\vec{B}(0)\right)$$

+ higher order

Torsion $= \tau = |B'|$

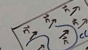
Homework 4.4.2

$C:(a,b) \to \mathbb{R}^3$ regular

lying in a plane $\boxed{P = \{\vec{x} : \vec{x}\cdot\vec{n} = C\}\ \text{where } \vec{n} \text{ is normal to } P}$

Show that whenever $T(t)$ and $N(t)$ are both defined they are parallel to P

$\boxed{\text{Must show: } \vec{T}\cdot\vec{n} = 0 \text{ and } \vec{N}\cdot\vec{n} = 0}$



P

whereever $C(t)$ goes $\vec{n}$ is the same

Since $\vec{C}(t) \in P$ we can say

$$\vec{C}(t)\cdot\vec{n} = C$$

Differentiate both sides:

$$\vec{C}'(t)\cdot\vec{n} + \vec{C}(t)\cdot\cancel{\tfrac{d}{dt}\vec{n}} = 0$$

0 because $\vec{n}$ is $\perp$ to P does not depend on time.

$$C'(t)\cdot\vec{n} = 0$$

So $\dfrac{C'(t)}{|C'(t)|}\cdot\vec{n} = 0$

So $T(t)\cdot\vec{n} = 0$

Next differentiate $T(t)\cdot\vec{n} = 0$ to find out about $T'(t)$ etc etc

4.4.4

Now show $\vec{B}(t) = \text{const}$ for the same curve.

So $\vec{T}(t)\cdot\vec{n} = 0$ and $\vec{N}(t)\cdot\vec{n} = 0$ from 4.4.2 solved.

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

So $\vec{B}(t)$ is $\perp$ to $\vec{T}(t)$ and $\vec{B}(t)$ is $\perp$ to $\vec{N}(t)$ and $|\vec{B}(t)| = 1$

So $\vec{B}$ is $\perp$ to plane spanned by $\vec{T}(t)$ + $\vec{N}(t)$

So $\vec{B} = \pm\vec{n}$ which is the normal to that plane.

So then const.