Teacher explanation notes · Corresponding question numbers match the student version. A hole remains excluded from the original domain even after cancellation.

1. Identify holes and vertical asymptotes. State the x-value of each.

$$f(x) = \frac{(x - 7)(x + 4)}{(x + 4)(x - 2)}$$

SOLUTION / EXPLANATION

Cancel $x + 4$. Original restrictions are $x \neq -4, 2$. Hole: $x = -4$, with $y = (-11)/(-6) = 11/6$. VA: $x = 2$.

Video guidance: 1, 3

2. Identify any holes and vertical asymptotes.

$$g(x) = \frac{x^2 - 16}{x^2 + 2x - 24}$$

SOLUTION / EXPLANATION

Factor $(x - 4)(x + 4) / [(x + 6)(x - 4)]$. Hole $x = 4$, $y = 8/10 = 4/5$; VA $x = -6$.

Video guidance: 1, 3

3. Identify all discontinuities and classify each as a hole or VA.

$$h(x) = \frac{(x - 3)^2(x + 1)}{(x - 3)(x + 5)}$$

SOLUTION / EXPLANATION

After canceling one $x - 3$, a hole remains at $x = 3$; its $y = (0)(4)/8 = 0$. VA $x = -5$.

Video guidance: 1, 3

4. Does this function have any holes or vertical asymptotes? Explain.

$$p(x) = \frac{x^2 + 4}{x^2 - 9}$$

SOLUTION / EXPLANATION

No common factors. The numerator is never zero for real x. Denominator $(x - 3)(x + 3)$ gives VAs $x = -3, 3$; no holes.

Video guidance: 1, 3

5. Write the left-hand and right-hand infinite limits at the vertical asymptote.

$f(x) =$

$$\frac{x + 2}{x - 5}$$

SOLUTION / EXPLANATION

At $x = 5$ numerator $7 > 0$. On the left denominator negative, so $\lim_{x \to 5^-} f(x) = -\infty$; right positive, so $\lim_{x \to 5^+} f(x) = +\infty$.

Video guidance: 3, 5

6. Write the left-hand and right-hand infinite limits at the vertical asymptote.

$g(x) =$

$$\frac{-3}{(x + 2)^2}$$

SOLUTION / EXPLANATION

At $x = -2$, denominator is positive and approaches 0 from both sides; negative numerator means both limits are $-\infty$.

Video guidance: 3, 5

7. Complete a calculator-style table near $x = 1$. Determine the VA and both one-sided limits.

$r(x) =$

$$\frac{x + 3}{x - 1}$$

| $x$ | 0.9 | 0.99 | 0.999 | 1 | 1.001 | 1.01 | 1.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | | |

SOLUTION / EXPLANATION

For x = 0.9, 0.99, 0.999, 1, 1.001, 1.01, 1.1: −39, −399, −3999, undefined, 4001, 401, 41. VA x = 1; left −∞ and right +∞.

Video guidance: 3, 5

8. Graph the rational function on the grid. State every vertical asymptote and its one-sided limits.

$f(x) = \dfrac{2}{(x + 3)(x - 2)}$

SOLUTION / EXPLANATION

VAs x = −3 and x = 2. At −3: left +∞, right −∞. At 2: left −∞, right +∞. Check signs of the two denominator factors.

Video guidance: 2, 4

9. Construct one rational function with a hole at x = −2 and vertical asymptotes at x = 1 and x = 5.

Use factored form.

SOLUTION / EXPLANATION

Example: [(x + 2)(x + 4)]/[(x + 2)(x − 1)(x − 5)]. The common x + 2 gives a hole, and uncanceled denominator zeros give VAs.

Video guidance: 2, 4

10. Construct one rational function with a hole at x = 4, a vertical asymptote at x = −1, and an x-intercept at x = 3.

Use factored form.

SOLUTION / EXPLANATION

Example: [(x − 4)(x − 3)]/[(x − 4)(x + 1)]. The zero at x = 3 remains, x = 4 cancels, x = −1 does not.

Video guidance: 2, 4

11. Write an equation that could match this graph: a hole at (−2, 2), vertical asymptote x = 1, and horizontal asymptote y = 3.

State a possible factored rational function.

SOLUTION / EXPLANATION

Use simplified y = 3 − 3/(x − 1) = 3(x − 2)/(x − 1); its value at −2 is 4, not 2. Instead y = 3 + 3/(x − 1) = 3x/(x − 1); at −2 gives 2. Include [(x + 2)3x]/[(x + 2)(x − 1)].

Video guidance: 2, 4

12. Analyze and sketch: VA, horizontal asymptote, intercepts, hole (coordinate), domain, and range.

$$s(x) = \frac{(x-4)(x+2)}{(x+2)(x-1)}$$

SOLUTION / EXPLANATION

Simplifies $(x-4)/(x-1) = 1 - 3/(x-1)$; VA $x = 1$; HA $y = 1$; x-int (4,0); y-int (0,4); hole at (−2,2); domain excludes −2,1; range excludes 1,2 ($y=2$ occurs only at excluded $x=-2$).

Video guidance: 2, 4

13. AP-style MCQ: For k(x) below, which statement is correct? A. Hole $x = 6$, VA $x = -4$ B. Hole $x = -4$, VA $x = 6$ C. Two holes D. Two VAs

$$k(x) = \frac{(x-6)(x+1)}{(x+4)(x-6)}$$

SOLUTION / EXPLANATION

A. $x - 6$ cancels and $x + 4$ remains in denominator.

Video guidance: 3, 5

14. AP-style MCQ: A rational function has a hole at $x = -3$ and a VA at $x = 2$. Which function works? A. $[[(x+3)/(x-2)]]$ B. $[[(x+3)(x-2)/(x+3)(x-5)]]$ C. $[[(x+3)(x+1)/(x+3)(x-2)]]$ D. $[[(x-2)/(x+3)]]$

Select A, B, C, or D.

SOLUTION / EXPLANATION

C. Cancel $x + 3$, leaving an uncanceled $x - 2$ in denominator.

Video guidance: 3, 5

15. Multiplicity: Which excluded x-values are holes, and which are VAs?

$$y = \frac{(x-1)^4(x+2)(x-5)^2}{(x-1)^2(x+2)^3(x-5)^2}$$

SOLUTION / EXPLANATION

x=1: denominator cancels completely ⇒ hole; x=−2: denominator still has power 2 ⇒ VA; x=5: fully cancels ⇒ hole.

Video guidance: 3, 5

16. Free response: Use x-values approaching 3 from both sides to explain the discontinuity, then state the horizontal asymptote and any hole.

$$f(x) = \frac{2(x + 1)(x - 5)}{(x - 3)(x - 5)}$$

SOLUTION / EXPLANATION

Cancel x−5 for x≠5: f=2(x+1)/(x−3). VA x=3, because factor remains. Hole at x=5 with y=2(6)/2=6. HA y=2. At $3^-$ f→−∞; at $3^+$ f→+∞.

Video guidance: 3, 5

## Teacher video resources

Videos 1–2 explain the ORIGINAL reference worksheets; use videos 3–5 to teach methods used in these newly numbered questions. They are not exact walkthroughs of the new numerical questions.

1. Original Worksheet A solution video
https://drive.google.com/file/d/189lexX0_DjSG2bvqGsJnz_uwExvoFDyx/view

2. Original Worksheet B solution video
https://drive.google.com/file/d/1_fFbd47Befsh6BUrcH2m7z3Ar_wl6yl2/view

3. Khan Academy: Discontinuities of rational functions
https://www.khanacademy.org/math/get-ready-for-ap-calc/xa350bf684c056c5c%3Aget-ready-for-limits-and-continuity/xa350bf684c056c5c%3Adiscontinuities-of-rational-functions/v/discontinuities-of-rational-functions

4. Khan Academy: Graphing according to asymptotes
https://www.khanacademy.org/math/algebra-2-fl-best/x727ff003d4fc3b92%3Arational-functions/x727ff003d4fc3b92%3Agraphs-of-rational-functions/v/finding-asymptotes-example

5. Graphing rational functions: holes, domain, range
https://www.youtube.com/watch?v=XE-22-F3oWw

## Simple teaching sequence

- 1. Factor numerator and denominator completely.

- 2. Record every excluded input from the original denominator.
- 3. Cancel common factors only to simplify expression; cancelled roots can be holes.
- 4. If a denominator factor remains, its root is a vertical asymptote.
- 5. Calculate the y-coordinate of a hole with the simplified expression.
- 6. For left/right limits, make a sign chart on each side of the asymptote.
- 7. For end behavior, compare leading terms. A horizontal asymptote need not be an excluded output unless shown by the range analysis.