

ប្រើមីត និងភាពជាប់លើលំហមេទ្រិក

នៃ (X, d) ជាលំហមេទ្រិក : $d : X \times X \rightarrow [0, +\infty)$

(1). ថា $d(x, y) = 0 \Leftrightarrow x = y$

(2). $d(x, y) = d(y, x)$, $\forall x, y$

(3). $d(x, z) \leq d(x, y) + d(y, z)$, $\forall x, y, z$

$(X, \|\cdot\|)$ ជាលំហ : $\|\cdot\| : X \times X \rightarrow [0, +\infty)$

(1). ថា $\|x\| = 0 \Leftrightarrow x = 0$

(2). $\forall x \in X, \forall \lambda \in \mathbb{K} (\mathbb{R}, \mathbb{C}) : \|\lambda x\| = |\lambda| \|x\|$

(3). $\forall x, y \in X, \|x + y\| \leq \|x\| + \|y\|$

គេដឹង $d = \|\cdot\|$ ជាមេទ្រិក

$d(x, y) = \|x - y\|$



$(\mathbb{R}, d)$, $d(x, y) = |x - y|$

$(\mathbb{R}, \|\cdot\|)$, $\|\cdot\| = |\cdot|$

$(\mathbb{R}^2, d)$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \|x - y\|$

$(X, d) = \text{complete} \Leftrightarrow$ គ្រប់ខ្លឹម Cauchy ប្រសើរ

• $(X, \|\cdot\|)$ = Banach $\Leftrightarrow$ complete

$\Leftrightarrow$ සෑම Cauchy දූම

• අර්ථය: $f: A \rightarrow B$ හි A හි සෑම x සඳහාම $f(x)$ B හි පවතී.

• විස්තෘතිය: $D_f = \{x \in A \mid f(x) \in B\}$

• අග්‍රය: $\text{Im} f = \{y \in B \mid \exists x \in A: f(x) = y\}$
 $= \{f(x) \mid x \in A\}$

• අග්‍රයේ ප්‍රතිබිම්බ: $f^{-1}(C) = \{x \in A \mid f(x) \in C\}$
 $C \subseteq B$

• සීමාපදය: (X, d) කිසිදු $a \in X$ සඳහාම (a_n) දූමක් a වෙත $a_n \rightarrow a$ වන පරිදි තිබේ.

$$\lim_{n \rightarrow \infty} a_n = a$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, d(a_n, a) < \varepsilon$$

Ex $(\mathbb{R}, d)$, $d = |\cdot|$ හි $(\mathbb{R}, |\cdot|)$

$$a_n = \frac{1}{n} \rightarrow 0$$

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, d\left(\frac{1}{n}, 0\right) < \varepsilon$$

$$\Leftrightarrow \left| \frac{1}{n} - 0 \right| < \varepsilon$$

$$\Leftrightarrow \frac{1}{n} < \varepsilon$$

$$\text{අපේක්ෂා } N \in \mathbb{N} : \frac{1}{N} < \varepsilon \Rightarrow N > \frac{1}{\varepsilon}$$

$$N = \left[\frac{1}{\varepsilon} \right] + 1$$

$$\forall n \geq N \Rightarrow \frac{1}{n} \leq \frac{1}{N} < \varepsilon \Rightarrow d\left(\frac{1}{n}, 0\right) < \varepsilon$$

$$\text{EX}_2: (\mathbb{R}^2, d = \|\cdot\|_2) \quad , \quad d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \|x - y\|_2$$

$\begin{matrix} (x_1, x_2) & (y_1, y_2) \end{matrix}$

$$a_n = \left(1 - \frac{1}{n}, \frac{1}{2n}\right) \longrightarrow (1, 0)$$

$$\begin{aligned} d(a_n, (1, 0)) &= \sqrt{\left(1 - \frac{1}{n} - 1\right)^2 + \left(\frac{1}{2n} - 0\right)^2} \\ &= \sqrt{\frac{1}{n^2} + \frac{1}{4n^2}} = \frac{1}{2n} \sqrt{5} \end{aligned}$$

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N \Rightarrow d(a_n, (1, 0)) < \varepsilon$$

$$\frac{1}{2n} \sqrt{5} \leq \frac{1}{2N} \sqrt{5}$$

$$\text{එනම් } \frac{1}{2N} \sqrt{5} < \varepsilon \Leftrightarrow N > \frac{\sqrt{5}}{2\varepsilon} \Rightarrow N = \left[\frac{\sqrt{5}}{2\varepsilon} \right] + 1$$

$$\forall n \geq N \Rightarrow d(a_n, (1, 0)) = \frac{1}{2n} \sqrt{5} < \frac{1}{2N} \sqrt{5} < \varepsilon$$

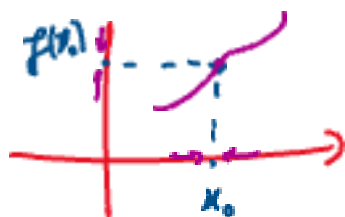
EX: $b_n = (\frac{1}{n}, z_{n+1})$ z_{n+1} is arbitrary

උදා: $(X, d_1), (Y, d_2)$ කාන්තරමයන් , $f: X \rightarrow Y$ අනුක්රමයක්
 යන f කාන්තරමයක් වනු ලබන්නේ $x \in X$



f finite (or) x_0 , f continuous , $\lim_{x \rightarrow x_0} f(x) = f(x_0)$
 $f(x) \neq \pm \infty$ $\lim_{x \rightarrow x_0} f(x) = L$ $x \neq x_0, L = f(x_0)$
 $\forall \epsilon > 0, \exists \delta > 0, \forall x: d_1(x, x_0) < \delta$
 $\Rightarrow d_2(f(x), L) < \epsilon$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x: d_1(x, x_0) < \delta \Rightarrow d_2(f(x), f(x_0)) < \varepsilon$$



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 12
 13

Ex: $(\mathbb{R}, |\cdot|)$, $X = Y = \mathbb{R}$, $f: \mathbb{R} \rightarrow \mathbb{R}$
 $x \mapsto f(x) = x^2$

$$x_0 = 2 \in \mathbb{R}, \quad f(2) = 4 \neq \infty$$

$$\forall \varepsilon > 0, \exists \delta > 0, \quad \forall x \in \mathbb{R}: d(x, 2) < \delta \Rightarrow d(x^2, 4) < \varepsilon$$

$$d(x^2, 4) = |x^2 - 4| = |(x-2)(x+2)| = d(x, 2) \cdot |x+2|$$

$$x \rightarrow 2 \quad \text{चुनकर } \delta \leq 1 \Leftrightarrow |x-2| < \delta \leq 1$$

$$\Rightarrow |x-2| < 1$$

$$-1 < x-2 < 1$$

$$3 < x+2 < 5$$

$$\Rightarrow |x+2| < 5$$

$$\text{अतः } \delta = \min \left\{ \frac{\varepsilon}{5}, 1 \right\}, \quad \forall x : d(x-2) < \delta \leq \frac{\varepsilon}{5}$$

$$\Rightarrow d(x^2, 4) = |x^2 - 4| = d(x, 2)|x+2| < \delta \cdot 5 \leq \frac{\varepsilon}{5} \cdot 5 = \varepsilon$$

$$\text{Ex: } f: (\mathbb{R}^2, d_1) \rightarrow (\mathbb{R}, d_2) \quad \text{चुनकर } (1, 2) \quad , \quad d_1 = \|\cdot\|_2 \\ (x, y) \rightarrow f(x, y) = x + 2y \quad d_2 = |\cdot|$$

$$d_2(f(x, y), f(1, 2)) = d_2(x + 2y, 5) = |x + 2y - 5|$$

$$d_1((x, y), (1, 2)) = \sqrt{(x-1)^2 + (y-2)^2} = \sqrt{x^2 - 2x + 1 + y^2 - 4y + 4} \\ = \sqrt{x^2 - 2x + y^2 - 4y + 5}$$

$$d_2(f(x, y), f(1, 2)) = |x + 2y - 5| = \sqrt{(x + 2y - 5)^2}$$



$$, \delta < 1, \quad \sqrt{(x-1)^2 + (y-2)^2} < \delta$$

$$|x-1| \leq \sqrt{(x-1)^2 + (y-2)^2} < \delta$$

$$|y-2| \leq \sqrt{(x-1)^2 + (y-2)^2} < \delta$$

$$d_2(f(x, y), f(1, 2)) = |x + 2y - 5| = |x-1 + 2y-4|$$

$$\leq |x-1| + |y-2| < \delta + 2\delta = 3\delta$$

$$\forall \varepsilon > 0, \text{ choose } \delta = \frac{\varepsilon}{3}, \forall (x,y) \in \mathbb{R} : d_1((x,y), (1,2)) < \delta$$

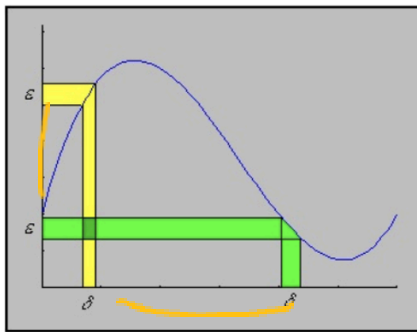
$$\Rightarrow d_2(f(x,y), f(1,2)) < 3\delta = 3 \cdot \frac{\varepsilon}{3} = \varepsilon$$

Def: $f: X \rightarrow Y$ is continuous $\Leftrightarrow f$ is continuous $\forall x \in X$

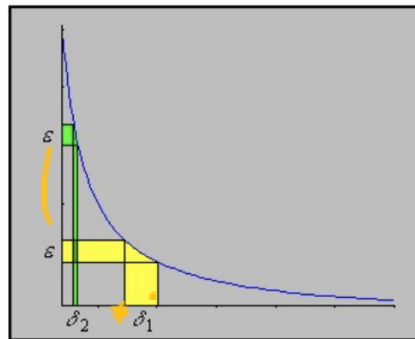
Def: $(X, d_1), (Y, d_2)$ are metric spaces, $f: X \rightarrow Y$

then f is uniformly continuous

$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in X : d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$$



(a) Uniformly Continuous



(b) Not Uniformly Continuous

Ex: $(\mathbb{R}, |\cdot|)$, $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous on $\mathbb{R}$
 $x \mapsto f(x) = 2x + 3$

$$|f(x) - f(y)| = |2x + 3 - 2y - 3| = 2|x - y|$$

$$\forall \varepsilon > 0, \text{ choose } \delta = \frac{\varepsilon}{2} \quad \forall x, y : |x - y| < \delta \Rightarrow |f(x) - f(y)| = 2|x - y| < 2\delta = \varepsilon$$

$$= 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

EX: $f: \mathbb{R} \rightarrow \mathbb{R}$ $x \rightarrow f(x) = x^2$ သပ် စစ်ချက်သပ် ကြ

$$= 1, \forall \delta > 0 \text{ ရှိ } x, y : |x - y| < \delta \text{ လျှင် } |f(x) - f(y)| \geq 1$$

$$x = n, y = n + \frac{\delta}{2}, n \in \mathbb{R}^+$$

$$|f(x) - f(y)| = |n^2 - (n + \frac{\delta}{2})^2| = |\delta n + \frac{\delta^2}{4}| = \delta n + \frac{\delta^2}{4} > 1, n \text{ သိပ်ကြီး}$$

$$\text{ပုံမှန် } n = \frac{1}{\delta} > 0, x = \frac{1}{\delta}, y = \frac{1}{\delta} + \frac{\delta}{2}$$

[နောက်ပိုင်း] $(x, d_x), (y, d_y)$ သာမန်မေထရစ်, $f: X \rightarrow Y$ သည်
 $(a_n) \subset X$ နှင့် $a \in X$ ကိုယ်စားပြု $f(a_n) \rightarrow f(a)$ ဖြစ်

ကျေးဇူးပြု:

f သည် a မှာ

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X: d_x(x, a) < \delta \Rightarrow d_y(f(x), f(a)) < \varepsilon$$

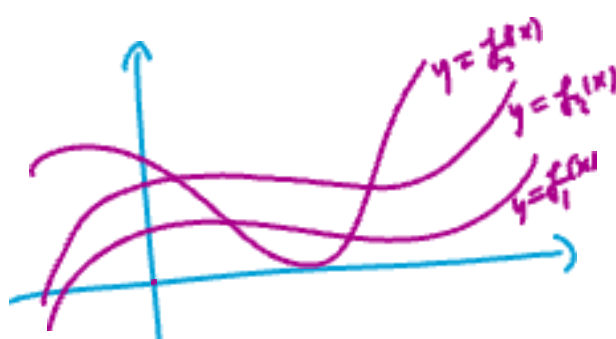
$$\text{အကယ်၍ } a_n \rightarrow a \Rightarrow \exists N \in \mathbb{N}, \forall n \geq N, d_x(a_n, a) < \delta$$

$$\Rightarrow d_y(f(a_n), f(a)) < \varepsilon$$

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N \Rightarrow d_y(f(a_n), f(a)) < \varepsilon$$



১) pointwise convergent (pointwise) and ২) uniform convergent (uniform convergent)
 ৩) series convergent



Def: A sequence (f_n) of functions $f_n: (X, d_x) \rightarrow (Y, d_y)$

is said to converge pointwise to a function f if

$$f_n(x) \rightarrow f(x), \quad x \in \mathbb{R} \quad \text{or} \quad f_n \rightarrow f$$

$$\Leftrightarrow \forall \varepsilon > 0, \forall x \in X, \exists N \in \mathbb{N} \Rightarrow d_y(f_n(x) - f(x)) < \varepsilon$$

(f_n) converges pointwise to f $\Rightarrow f_n \Rightarrow f$

$$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall x \in X, d_y(f_n(x) - f(x)) < \varepsilon$$

3: $f_n: [0, 1] \rightarrow \mathbb{R}, \quad f_n(x) = x^n, \quad d = |\cdot|$

• $x = 1 \Rightarrow f_n(1) = 1 \rightarrow 1 \Leftrightarrow \forall \varepsilon > 0, \exists N = 1, \forall n \geq 1, |1 - 1| = 0 < \varepsilon$

• $x < 1 \Rightarrow f_n(x) = x^n \rightarrow 0 \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x^n - 0| < \varepsilon$

$x^n < x^N < \varepsilon \Rightarrow x^N < \varepsilon$

$\text{If } \varepsilon > 1 \Rightarrow N = 1$
 $\Rightarrow x^N \leq x^N < 1 + \varepsilon$
 $\text{If } \varepsilon < 1 \Rightarrow N \ln x < \ln \varepsilon$
 $N > \frac{\ln \varepsilon}{\ln x}$

$\text{Choose } N = \max \left\{ 1, \frac{\ln \varepsilon}{\ln x} \right\}$



$\Rightarrow f_n(x) \xrightarrow{x^n} f(x) = \begin{cases} 1, & x = 1 \\ 0, & 0 \leq x < 1 \end{cases}$

$\text{Does } f_n \Rightarrow f \text{ u.s. ?}$

$|f_n(x) - f(x)| = \begin{cases} |f_n(x) - 1| & x = 1 \\ |f_n(x)| & 0 \leq x < 1 \end{cases}$

$x \text{ close to } 1, \quad x = 1 - \frac{1}{n} \Rightarrow |f_n(x) - 1| = \left| \left(1 - \frac{1}{n}\right)^n - 1 \right| \approx |1 - e^{-1}|$
 $|f_n(x)| = \left| \left(1 - \frac{1}{n}\right)^n \right| \rightarrow e^{-1}$

$f_n \Rightarrow f \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall x \in X, \underbrace{d_Y(f_n(x), f(x))}_{\sup_{x \in X} d_Y(f_n(x), f(x))} < \varepsilon$

$\wedge \sup_{x \in X} d_Y(f_n(x), f(x)) \rightarrow 0$

Ex: $f_n : (0,1) \rightarrow \mathbb{R}, f_n(x) = x^n \xrightarrow{n \rightarrow \infty} 0$

$\sup_{x \in (0,1)} |x^n - 0| < \left(\frac{1}{2}\right)^n \rightarrow 0$

$$f_n: [0,1] \rightarrow \mathbb{R} : f_n(x) = \frac{1}{n(x+1)} \rightarrow f(x) = \begin{cases} 1, & x=0 \\ 0, & x \neq 0 \end{cases}$$

$$\sup_{x \in [0,1]} \left| \frac{1}{n(x+1)} - f(x) \right| = 1 \not\rightarrow 0$$

$$0 < r < 1, f_n: [r,1] \rightarrow \mathbb{R} : f_n(x) = x^n \rightarrow 0$$

Example: $f_n: [0,1] \rightarrow \mathbb{R} : f_n(x) = \frac{n^3 x}{e^{nx}}$ (pointwise limit, $\lim_{n \rightarrow \infty} f_n(x) = 0$)

$$f_n: [0,1] \rightarrow \mathbb{R} : f_n(x) = \frac{n^3 x}{e^{nx}}, \quad d = 1.1.$$

$$\cdot x=0 \Rightarrow f_n(0) = 0 \rightarrow 0$$

$$\cdot x=1 \Rightarrow f_n(1) = \frac{n^3}{e^n} \rightarrow 0$$

$$\cdot 0 < x < 1 \Rightarrow f_n(x) = \frac{n^3 x}{e^{nx}} < \frac{n^3}{e^{nx}} \rightarrow 0$$

$$\Rightarrow f_n(x) \rightarrow f(x) = 0$$

$$\sup_{x \in [0,1]} |f_n(x) - 0| = ?$$

$$x \in [0,1]$$

$$g_n(x) = |f_n(x) - 0| = f_n(x) = \frac{n^3 x}{e^{nx}} \Rightarrow f'_n(x) = \frac{n^3 e^{nx} - n e^{nx} \cdot n^3 x}{e^{2nx}}$$

$$g'_n(x) = f'_n(x) = 0 \Leftrightarrow 1 - nx = 0$$

$$x = \frac{1}{n}$$



$$\sup_{x \in [0,1]} |f_n(x) - 0| = g_n\left(\frac{1}{n}\right) = \frac{n^3}{e^{n \cdot \frac{1}{n}}} \cdot \frac{1}{n} = \frac{n^2}{e} \rightarrow \infty$$

$\Rightarrow \nrightarrow f$ (uniformly). $\square$

उदाहरण. f_n का प्रत्येक $n \in \mathbb{N}$ के लिए $f_n \Rightarrow f$ का कोई नहीं है

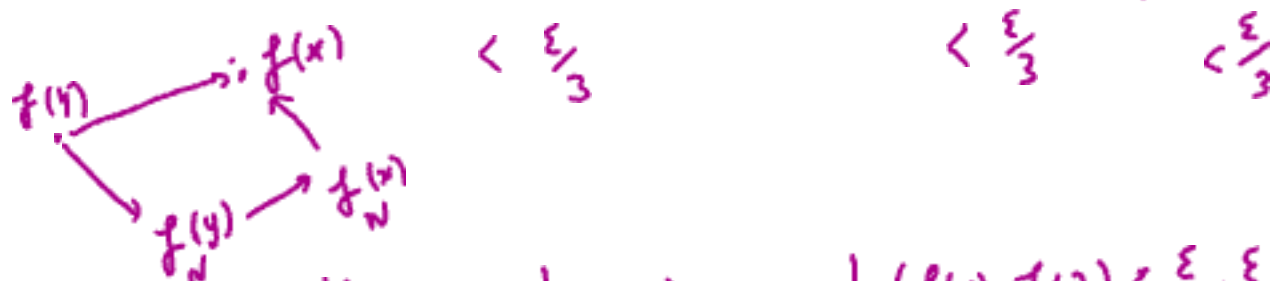
उदाहरण. $f_n(x) = x^n$ का प्रत्येक $[0,1]$ पर $f(x) = \begin{cases} 1, & x=1 \\ 0, & x < 1 \end{cases}$

का प्रत्येक $f_n(x) = \frac{n^3 x}{nx}$ का प्रत्येक $[0,1]$ पर $f(x) = 0$ का प्रत्येक

किसी $\epsilon > 0$, $\exists N \in \mathbb{N}$, $\forall x \in X$, $d_Y(f_n(x), f(x)) < \frac{\epsilon}{3}$, $\forall n \geq N$

$f_n: X \rightarrow Y$, f_n का प्रत्येक $\Rightarrow f_N$ का प्रत्येक $\Leftrightarrow \exists \delta > 0$, $\forall y \in X$; $d_X(x, y) < \delta$
 $\Rightarrow d_Y(f_N(y), f_N(x)) < \frac{\epsilon}{3}$

$$d_Y(f(y), f(x)) \leq d_Y(f(y), f_N(y)) + d_Y(f_N(y), f_N(x)) + d_Y(f_N(x), f(x))$$



$$\forall \epsilon > 0, \exists \delta > 0, \forall y \in X: d_X(x, y) < \delta \Rightarrow d_Y(f(y), f(x)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

$$\Leftrightarrow d_Y(f(y), f(x)) < \epsilon \quad \square$$

Num 1: $f: \mathbb{R} \rightarrow \mathbb{R}$ such, $f(1) \neq 0$ and

$$\forall x, y \in \mathbb{R}: f(x+y) = f(x) + f(y)$$

Ans: $\mathbb{R} \cong \text{Homomorphism}$

$$y = ax + b$$

$$(\mathbb{R}, +) = \text{group}$$

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c, \quad \mathbb{Q} = \mathbb{Z} \cup \mathbb{Z}/\mathbb{Z}_{\neq 0}$$

$$\mathbb{Z} = \mathbb{N} \cup (-\mathbb{N})$$

$$\cdot \mathbb{N} \rightarrow f(x) = ?$$

$$\cdot \mathbb{Z} \rightarrow f(x) = ?$$

$$\cdot \mathbb{Q} \rightarrow f(x) = ?$$

$$\cdot \mathbb{R} \rightarrow f(x) = ?$$

$$\cdot \forall x = y \Rightarrow f(2x) = 2f(x), \quad f(0) = 2f(0) \Rightarrow f(0) = 0$$

$$x = 1 \Rightarrow f(2) = 2f(1), \quad f(1) \neq 0$$

$$x = 3 \Rightarrow f(3) = f(2+1) = f(2) + f(1) = 3f(1)$$

$$\forall x \in \mathbb{N} \Rightarrow f(x) = x \cdot f(1)$$

$$x = 2, \quad f(2) = 2 \cdot f(1)$$

$$\exists \forall m \in \mathbb{N}; x \in \mathbb{N}, x \geq 2 \text{ then } \exists \text{ even}$$

$$\Rightarrow x+1 \in \mathbb{N} \Rightarrow f(x+1) = f(x) + f(1)$$

$$= x f(1) + f(1)$$

$$= (x+1) f(1)$$

$$\text{ಎಸ } a = f(1) \neq 0 \Rightarrow \forall x \in \mathbb{N} : f(x) = ax$$

$$x \in -\mathbb{N} \Rightarrow x < 0 \Rightarrow -x \in \mathbb{N}$$

$$\Rightarrow f(x) = a(-x)$$

$$\text{ಹ } \begin{cases} f(x-x) = f(0) = 0 \\ f(x-x) = f(x+(-x)) = f(x) + f(-x) \end{cases}$$

$$\Rightarrow f(-x) = -f(x)$$

$$\Rightarrow -f(x) = a(-x) \Rightarrow f(x) = ax$$

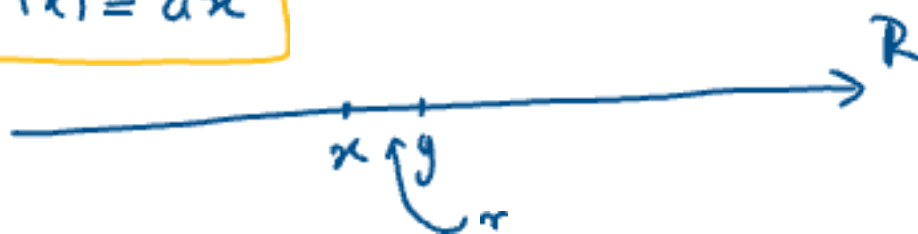
$$\Rightarrow \forall x \in \mathbb{Z} : f(x) = ax$$

$$\underline{\underline{x = \frac{n}{m} \in \mathbb{Q}, m \neq 0}} \Rightarrow f(x) = f\left(\frac{n}{m}\right) = f\left(\underbrace{\frac{1}{m} + \frac{1}{m} + \dots + \frac{1}{m}}_{n \text{ times}}\right)$$

$$\text{ಹ } f(1) = f\left(\frac{m}{m}\right) = m f\left(\frac{1}{m}\right) \Rightarrow f\left(\frac{1}{m}\right) = \frac{1}{m} f(1) = \frac{1}{m} a$$

$$\Rightarrow f(x) = f\left(\frac{n}{m}\right) = n f\left(\frac{1}{m}\right) = \frac{n}{m} f(1) = x f(1) = ax$$

$$\boxed{\forall x \in \mathbb{Q} : f(x) = ax}$$



ಎರಡನೆಯ ಮಹಾಸೂತ್ರ (R, |·|), $\overline{\mathbb{Q}} = \mathbb{R}$

$$\bar{\mathbb{Q}} = \mathbb{Q}' \cup \mathbb{Q} \Rightarrow \mathbb{Q}' \supseteq \mathbb{Q}^c = \text{irrational number.}$$

$$\Rightarrow \mathbb{Q}^c \subseteq \bar{\mathbb{Q}} = \text{සමස්ත සංඛ්‍යා}$$

$$r \in \mathbb{Q}^c \Rightarrow \exists x_n = \frac{[10^n r]}{10^n} \longrightarrow r \in \mathbb{Q}^c$$

$$\boxed{\forall r \in \mathbb{Q}^c} \Rightarrow \exists (x_n) \subseteq \mathbb{Q} : x_n \xrightarrow{\mathbb{Q}} r$$

$$\text{if } f \text{ is continuous} \Rightarrow f(x_n) \longrightarrow f(r)$$

$$f(x_n) = ax_n \Rightarrow ax_n \longrightarrow ar \quad (\text{limit of } ax_n)$$

$$\text{මෙම අවස්ථාවේදී } f(r) = ar \Rightarrow \boxed{f(r) = ar}$$

$$\therefore \forall x \in \mathbb{R} : f(x) = ax \quad \text{සාධක} \quad \square$$

Def: $\sum_{k=0}^{\infty} a_k = \sum a_n = a_0 + a_1 + a_2 + a_3 + \dots$

$\sum_{k=1}^{\infty} a_k = \sum a_n = a_1 + a_2 + a_3 + \dots$

ආදායම්: $\sum_{k=0}^{\infty} f_k(x) = f_0(x) + f_1(x) + f_2(x) + \dots$

$$\sum_{k=1}^{\infty} f_k(x) = f_1(x) + f_2(x) + f_3(x) + \dots$$

$$\therefore \text{if } S_n(x) = \sum_{k=1}^n f_k(x) = f_1(x) + f_2(x) + \dots + f_n(x)$$

$$\sum_{k=1}^{\infty} f_k(x) \text{ converges} \Leftrightarrow (S_n(x)) \text{ converges}$$

$\sum_{n=1}^{\infty} f_n$ exists $\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall x \in X,$
 $\forall m > n \geq N, d_Y(\sum_{k=1}^m f_k(x), \sum_{k=1}^n f_k(x)) < \varepsilon$

$$\left\| \sum_{k=1}^m f_k(x) - \sum_{k=1}^n f_k(x) \right\| < \varepsilon$$

$$\left\| \sum_{k=n+1}^m f_k(x) \right\| < \varepsilon$$

space X complete (Banach), $\Rightarrow$ Cauchy $\Rightarrow$ exists

$\Rightarrow$ Cauchy $\Rightarrow$ exists

Weierstrass-M-test:

$f_n: A \rightarrow \mathbb{R}, A \subseteq \mathbb{R}$ such that $\forall n \in \mathbb{N}, \exists M_n > 0$

such that $|f_n(x)| \leq M_n, \forall x \in A$ and $\sum M_n < \infty$

$\Rightarrow \sum_{n=1}^{\infty} f_n(x)$ exists

Proof: $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m > n \geq N \Rightarrow \left| \sum_{k=n+1}^m M_k \right| < \varepsilon$

$\forall x \in A, \forall m > n \geq N$

$$\Rightarrow \left| \sum_{k=n+1}^m f_k(x) \right| \leq \sum_{k=n+1}^m |f_k(x)| \leq \sum_{k=n+1}^m M_k < \varepsilon$$

$\therefore \sum f_n$ converges ✓

EX: $f_n(x) = x^n$, $\forall x \in [-r, r]$, $0 < r < 1$

$$|f_n(x)| = |x^n| < r^n$$

$$\Rightarrow \sum r^n = r + r + r^2 + \dots = \frac{1}{1-r} < \infty$$

$\therefore$ Weierstrass M-test: $\sum x^n$ converges ✓

EX: $f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cos(3^n x)$ converges (M-test):

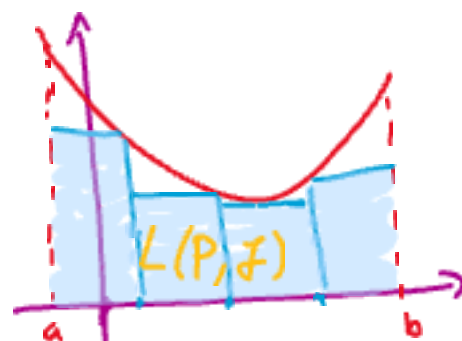
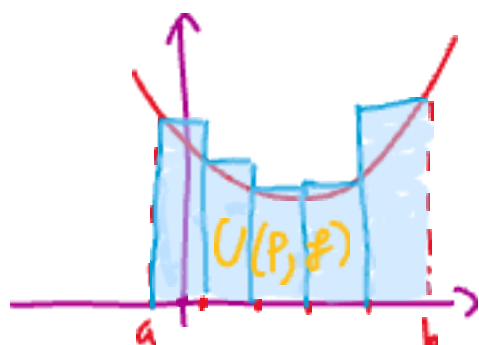
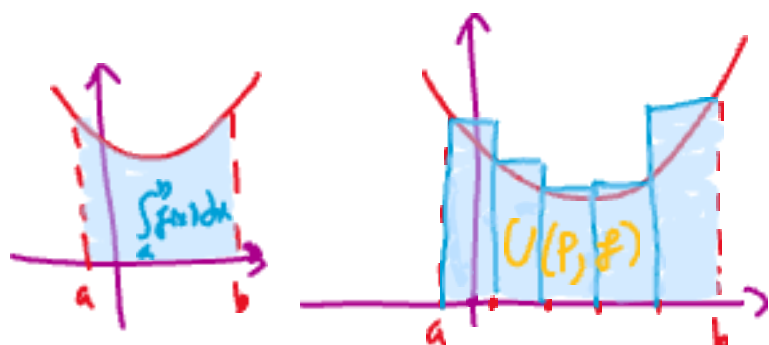
$$\text{and } f_n(x) = \frac{1}{2^n} \cos(3^n x)$$

$$\forall n \in \mathbb{N}, |f_n(x)| = \left| \frac{1}{2^n} \cos(3^n x) \right| \leq \frac{1}{2^n} = M_n$$

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2 < \infty$$

$\therefore$ Weierstrass M-test: $\sum_{n=1}^{\infty} \frac{1}{2^n} \cos(3^n x)$ converges

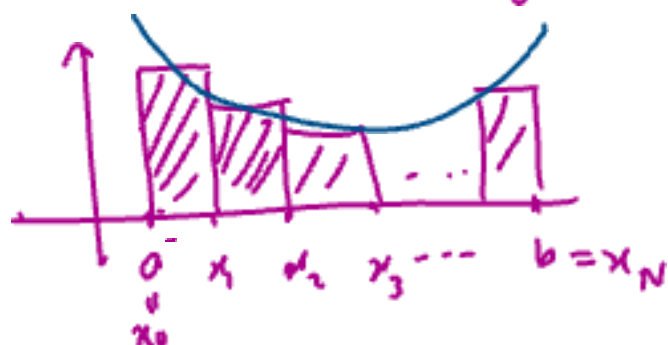
Another way Riemann:



$$A = \int_a^b f(x) dx, \quad U(P, f) = \text{အလယ်ပြင်ရွက်တဲ့ စတုရန်းကလေးများ၏ ပေါက်ပုံစံ},$$

$$L(P, f) = \text{အလယ်ပြင်ရွက်တဲ့ စတုရန်းကလေးများ၏ အောက်ပေါက်ပုံစံ}$$

Def: P သာမန်ပုံစံ ကို $[a, b]$ သို့မဟုတ် $\{x_0, x_1, x_2, \dots, x_N\}$
 $x_0 = a, x_N = b$



$$U(P, f) = \sum_{i=1}^N M_i (x_i - x_{i-1}), \quad M_i = \sup \{f(x) : x_{i-1} \leq x \leq x_i\}$$

$$L(P, f) = \sum_{i=1}^N m_i (x_i - x_{i-1}), \quad m_i = \inf \{f(x) : x_{i-1} \leq x \leq x_i\}$$

$$\Rightarrow L(P, f) \leq \int_a^b f(x) dx \leq U(P, f) \quad \square$$

$$m = \min \{m_i\}, \quad M = \max \{M_i\}$$

$$\Rightarrow \sum_{i=1}^N m (x_i - x_{i-1}) \leq \sum_{i=1}^N m_i (x_i - x_{i-1}) \leq \sum_{i=1}^N M_i (x_i - x_{i-1})$$

$$\leq \sum_{i=1}^N M (x_i - x_{i-1})$$

$$m(b-a) \leq L(P, f) \leq A \leq U(P, f) \leq M(b-a)$$

$$\overline{\int_a^b f(x) dx} = U \int_a^b f(x) dx = \inf \{ U(P, f) : P \text{ partition of } [a, b] \}$$

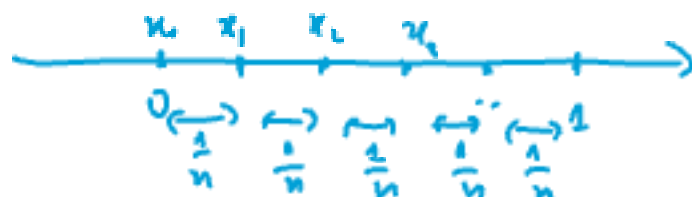
$$\underline{\int_a^b f(x) dx} = L \int_a^b f(x) dx = \sup \{ L(P, f) : P \text{ partition of } [a, b] \}$$

$$\Rightarrow \overline{\int_a^b f(x) dx} = \underline{\int_a^b f(x) dx} = \int_a^b f(x) dx$$

EX: $f(x) = 2x + 1$ on $[0, 1]$ we know $\int_0^1 f(x) dx$ is Riemann

$$P_n = \{x_0 = 0, x_1, x_2, \dots, x_n = 1\}$$

EX: $x_i - x_{i-1} = \frac{1-0}{n} = \frac{1}{n}, x_i = 0 + i \frac{1}{n} = \frac{i}{n}$



$$M_i = \sup \{ f(x) : x_{i-1} \leq x \leq x_i \} = 2x_{i-1} + 1 = 2\left(\frac{i-1}{n}\right) + 1$$

$$m_i = \inf \{ f(x) : x_{i-1} \leq x \leq x_i \} = 2x_i + 1 = 2\frac{i}{n} + 1$$

$$U(P_n, f) = \sum_{i=1}^n \left(2\left(\frac{i-1}{n}\right) + 1 \right) \cdot \frac{1}{n} \quad \text{and} \quad \rightarrow \inf (U(P_n, f) : P_n \text{ partition of } [0, 1])$$

$$L(P_n, f) = \sum_{i=1}^n \left(2\frac{i}{n} + 1 \right) \frac{1}{n} \quad \text{and} \quad \rightarrow \sup (L(P_n, f) : P_n \text{ partition of } [0, 1])$$

$$\Rightarrow \overline{\int_0^1 (2x+1) dx} = \inf_{\forall P_n} (U(P_n, f)) = \lim_{n \rightarrow \infty} U(P_n, f)$$

$$\underline{\int_a^b (2x+1) dx} = \sup_{\forall P_n} (L(P_n, f)) = \lim_{n \rightarrow \infty} L(P_n, f)$$

$$U(P_n, f) = \sum_{i=1}^n \left(\frac{2}{n^2} i - \frac{2}{n^2} + \frac{1}{n} \right) = \frac{2}{n^2} \sum_{i=1}^n i - \sum_{i=1}^n \frac{2}{n^2} + \sum_{i=1}^n \frac{1}{n}$$

$$= \frac{2}{n^2} \cdot \frac{n(n+1)}{2} - \frac{2n}{n^2} + \frac{n}{n}$$

$$\lim_{n \rightarrow \infty} U(P_n, f) = 1 + 1 = 2$$

$$L(P_n, f) = \sum_{i=1}^n \left(\frac{2}{n^2} i + \frac{1}{n} \right) = \frac{2}{n^2} \frac{n(n+1)}{2} + \frac{n}{n}$$

$$\lim_{n \rightarrow \infty} L(P_n, f) = 2$$

$$\Rightarrow \int_0^1 (2x+1) dx = 2 \quad \square$$

ନିଉଟ: $f(x) = x^2$, $[-1, 1]$ କର $\int_{-1}^1 f(x) dx$ ର Riemann.

(Homework)

ଉଦା: f_n କ୍ରମିକ ଭାବେ ନିମ୍ନ: $[a, b]$ ଓ ଯେ $f_n \Rightarrow f$

ନିମ୍ନ: $[a, b]$, f_n ର Riemann $[a, b]$

$\Rightarrow f$ ର Riemann $[a, b]$, $\int_a^b f_n \rightarrow \int_a^b f$

Conv. $f_n \Rightarrow f \Rightarrow \forall n \in \mathbb{N}, \exists \alpha_n > 0: |f_n(x) - f(x)| \leq \alpha_n \rightarrow 0$
 $\Rightarrow -\alpha_n \leq f_n(x) - f(x) \leq \alpha_n$

$$-f_n(x) - \alpha_n \leq -f(x) \leq \alpha_n - f_n(x)$$

$$f_n(x) - \alpha_n \leq f(x) \leq \alpha_n + f_n(x)$$

$$L(P_n, f_n - \alpha_n) \leq L(P_n, f) \leq U(P_n, f_n) \leq U(P_n, \alpha_n + f_n)$$

$$L(P_n, f_n) - L(\alpha_n) \leq L(P_n, f) \leq U(P_n, f_n) \leq U(P_n, \alpha_n) + U(P_n, f_n)$$

$$\int_a^b f_n(x) dx - \alpha_n(b-a) \leq L(P_n, f) \leq U(P_n, f) \leq 2(b-a) + \int_a^b f_n(x) dx$$

$$\Rightarrow \left| \int_a^b f_n(x) dx - \int_a^b f(x) dx \right| \leq \alpha_n(b-a) \rightarrow 0$$

$\rightarrow f$ Riemann-integrabel (oder Riemann)

$$\int_a^b f_n(x) dx \rightarrow \int_a^b f(x)$$



EX: $f_n(x) = x^n \Rightarrow 0$ mit $\alpha_n: [-r, r], 0 \leq r < 1$
 $\int_{-r}^r x^n dx \rightarrow \int_{-r}^r 0 dx = 0$