1.

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

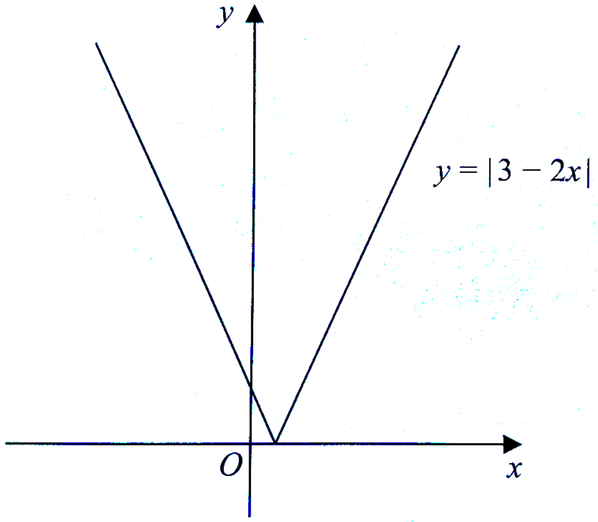


Figure 1

Figure 1 shows a sketch of the graph with equation $y = |3 - 2x|$

Solve

$$|3 - 2x| = 7 + x$$

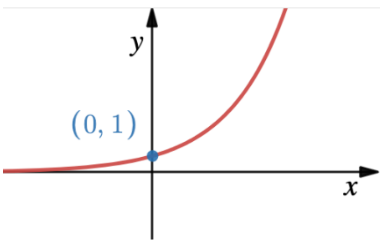
$$\begin{aligned} 3 - 2x &= 7 + x \\ -4 &= 3x \\ x &= -\frac{4}{3} \\ 2x - 3 &= 7 + x \\ x &= 10 \end{aligned}$$

(4)

2.

a. Sketch the curve with equation

$$y = 4^x$$



(2)

b. Solve

$$4^x = 100$$

giving your answer to 2 decimal places.

$$\begin{aligned} x &= \log_4 100 \\ &= 3.32\,(2\text{dp}) \end{aligned}$$

(2)

3.

A sequence of terms $a_1,\ a_2,\ a_3, \ldots$ is defined by

$$a_1 = 3$$
$$a_{n+1} = 8 - a_n$$

a.

i. Show that this sequence is periodic

$$a_2 = 8 - 3 = 5$$
$$a_3 = 8 - 5 = 3 = a_1$$

ii. State the order of this periodic sequence.

3,5,3,5... period of 2

(2)

b. Find the value of

$$\sum_{n=1}^{85} a_n$$

$$\sum_{n=1}^{85} a_n = 42 \times (3+5) + 3 = 339$$

(2)

4.

Given that

$$y = 2x^2$$

Use differentiation to from first principles to show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{h\to 0} \frac{2(x+h)^2 + 2x^2}{h} \\ &= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 - 2x^2}{h} \\ &= \lim_{h\to 0} 4x + 2h \\ &= 4x\end{aligned}$$

(3)

5.

The table below shows corresponding values of x and y for $y = \log_3 2x$

The values of y are given to 2 decimal places as appropriate.

x	3	4.5	6	7.5	9
y	1.63	2	2.26	2.46	2.63

a. Using the trapezium rule with all the values of y in the table, find an estimate for

$$\int_3^9 \log_3 2x \, \mathrm{d}x$$

$$\begin{aligned}\int_3^9 \log_3 2x \, \mathrm{d}x &= \frac{1}{2}(1.5)[1.63 + 2.63 + 2(2 + 2.26 + 2.46)] \\ &= 13.3 \, (3\text{sf})\end{aligned}$$

(3)

Using your answer to part (a) and making your method clear, estimate

b.

i. $\int_3^9 \log_3 (2x)^{10} \mathrm{d}x$

$$\begin{aligned}\int_3^9 \log_3 (2x)^{10} \mathrm{d}x &= 10 \int_3^9 \log_3 (2x) \mathrm{d}x \\ &= 133\end{aligned}$$

ii. $\int_3^9 \log_3 18x \, \mathrm{d}x$

$$\begin{aligned}\int_3^9 \log_3 18x \, \mathrm{d}x &= \int_3^9 \log_3 9 \times 2x \, \mathrm{d}x \\ &= \int_3^9 9 \, \mathrm{d}x + \int_3^9 \log_3 2x \, \mathrm{d}x \\ &= [9x]_3^9 + 13.3 \\ &= 67.3\end{aligned}$$

(3)

6.

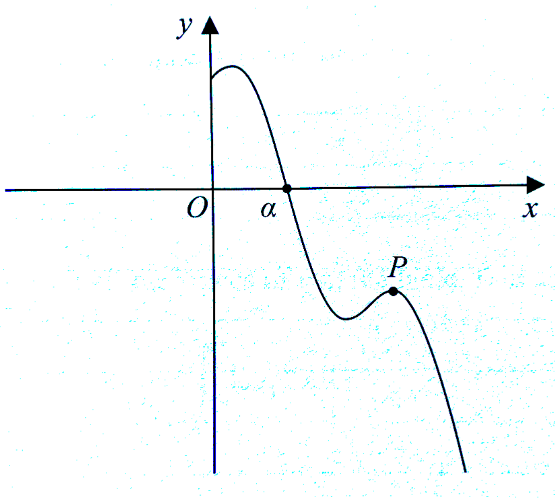


Figure 2

Figure 2 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 8\sin\left(\frac{1}{2}x\right) - 3x + 9 \qquad x > 0$$

and x is measured in radians.

The point P, shown in Figure 2, is a local maximum point on the curve.

Using calculus and the sketch in Figure 2,

a. Find the x coordinates of P, giving your answer to 3 significant figures.

$$\mathrm{f}'(x) = 4\cos\left(\frac{1}{2}x\right) - 3 = 0$$

$$\cos\left(\frac{1}{2}x\right) = \frac{3}{4}$$

$$\frac{1}{2}x = 0.723,\ 5.56 + 2\pi$$

$$x = 1.44,\ 11.1 + 4\pi$$

$$x \text{ coordinates of } P = 14.0$$

(4)

The curve crosses the x-axis at $x = \alpha$, shown in Figure 2.

Given that, to 3 decimal places, $\mathrm{f}(4) = 4.274$ and $\mathrm{f}(5) = -1.212$

b. Explain why α must lie in the interval $[4, 5]$

Changes signs and $\mathrm{f}(x)$ is continuous therefore $\mathrm{f}(x) = 0$ has a root lie in the interval.

(1)

c. Taking $x_0 = 5$ as a first approximation to α, apply the Newton-Raphson method once to $\mathrm{f}(x)$ to obtain a second approximation to α.

Show your method and give your answer to 3 significant figures.

$$x_1 = x_0 - \frac{\mathrm{f}(x_0)}{\mathrm{f}'(x_0)}$$

$$= 5 - \frac{8\sin\frac{5}{2} - 15 + 9}{4\cos\frac{5}{2} - 3}$$

$$= 4.80\ (3\text{sf})$$

(2)

7.

a. Find the first four terms, in ascending powers of x, of the binomial expansion of

$$\sqrt{4-9x}$$

writing each term in simplest form.

$$\begin{aligned}\sqrt{4-9x} &= 4^{\frac{1}{2}}\left(1-\frac{9}{4}x\right)^{\frac{1}{2}} \\ &= 2\left[1+\left(\frac{1}{2}\right)\left(-\frac{9}{4}x\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\left(-\frac{9}{4}x\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\left(-\frac{9}{4}x\right)^3\right] \\ &= 2-\frac{9}{4}x-\frac{81}{64}x^2-\frac{729}{512}x^3\end{aligned}$$

(4)

A student uses this expansion with $x=\frac{1}{9}$ to find an approximation for $\sqrt{3}$

Using the answer to part (a) and without doing any calculations,

b. State whether this approximation will be an overestimate or an underestimate of $\sqrt{3}$ giving a brief reason for your answer.

Overestimate, as the next term of the series is negative, more will give a more accurate approximation.

(1)

8.

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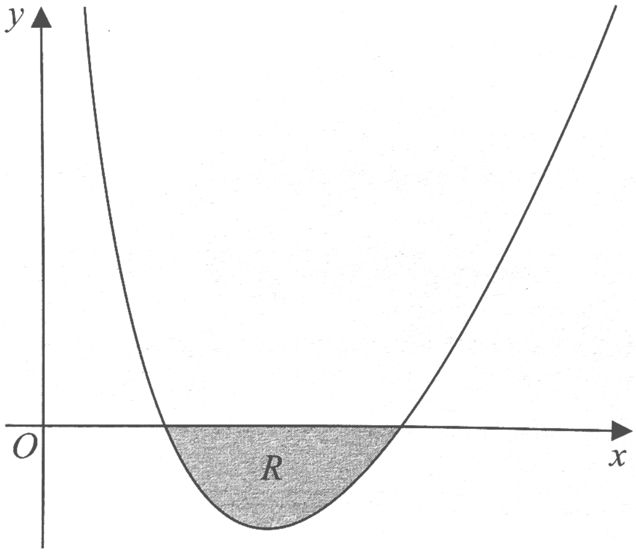

Figure 3

Figure 3 shows a sketch of part of a curve with equation

$$y = \frac{(x-2)(x-4)}{4\sqrt{x}} \qquad x > 0$$

The region *R*, shown shaded in Figure 3, is bounded by the curve and the x-axis.

Find the exact area of *R*, writing your answer in the form $a\sqrt{2}+b$, where *a* and *b* are constants to be found.

$$\begin{aligned}
I &= \int_2^4 \frac{(x-2)(x-4)}{4\sqrt{x}}\,\mathrm{d}x \\
&= \frac{1}{4}\int_2^4 \frac{x^2-6x+8}{x^{\frac{1}{2}}}\,\mathrm{d}x \\
&= \frac{1}{4}\int_2^4 \left(x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 8x^{-\frac{1}{2}}\right)\mathrm{d}x \\
&= \frac{1}{4}\left[\frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{3}{2}} + 16x^{\frac{1}{2}}\right]_2^4 \\
&= \frac{1}{4}\left[\frac{2}{5}\cdot 32 - 4\cdot 8 + 16\cdot 2 - \left(\frac{2}{5}\cdot 4\sqrt{2} - 4\cdot 2\sqrt{2} + 16\sqrt{2}\right)\right] \\
&= \frac{1}{4}\left(\frac{64}{5} - \frac{48}{5}\sqrt{2}\right) \\
&= \frac{32}{5} - \frac{24}{5}\sqrt{2} \\
\text{area} &= \frac{24}{5}\sqrt{2} - \frac{32}{5} \\
a &= \frac{32}{5},\ b = -\frac{24}{5}
\end{aligned}$$

(6)

9.

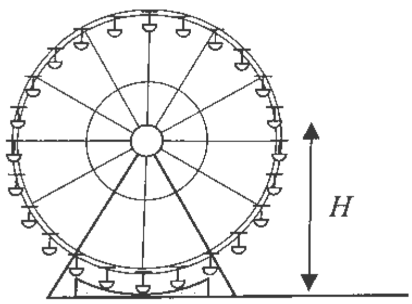

Figure 4

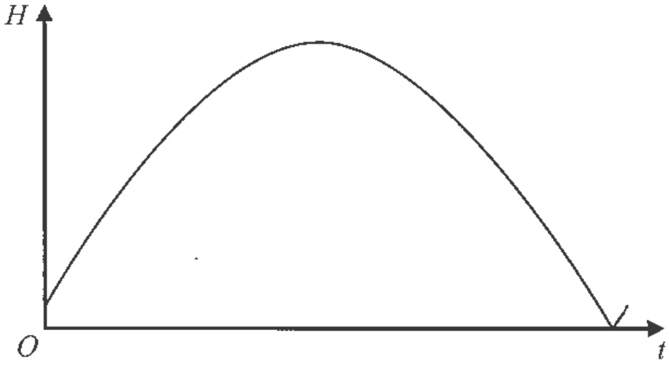

Figure 5

Figure 4 shows a sketch of a Ferris wheel.

The height above the ground, H m, of a passenger on the Ferris wheel, t seconds after the wheel starts turning, is modelled by the equation

$$H = |A\sin(bt+\alpha)^\circ|$$

where A, b and α are constant.

Figure 5 shows a sketch of the graph of H against t, for one revolution of the wheel.

Given that

- the maximum height of the passenger above the ground is 50 m
- the passenger is 1 m above the ground when the wheel starts turning
- the wheel takes 720 seconds to complete one revolution

a. find a complete equation for the model, giving the exact value of A, the exact value of b and the value of α to 3 significant figures.

$$\text{when } \sin(bt+\alpha)=1,\ A=50$$

$$\text{when } \sin(bt+\alpha)=0$$

$$bt+\alpha = 0, 180+360n$$

$$t = \frac{-\alpha}{b}, \frac{180-\alpha}{b} + \frac{360n}{b}$$

$$\frac{360}{b} = 2\times 720$$

$$b = \frac{1}{4}$$

$$\text{when } t=0,\ H=1$$

$$1 = 50\sin(\alpha)$$

$$\alpha = 1.16^\circ \text{ (3sf)}$$

$$H = \left|50\sin\left(\frac{t}{4}+1.16\right)^\circ\right|$$

(4)

b. Explain why an equation of the form

$$H = |A\sin(bt+\alpha)^\circ| + d$$

where d is a positive constant, would be a more appropriate model.

The platform when entering the carriage has some height.

(1)

10.

The function f is defined by

$$\mathrm{f}(x) = \frac{8x+5}{2x+3} \qquad x > -\frac{3}{2}$$

a. Find $\mathrm{f}^{-1}\left(\frac{3}{2}\right)$

$$\begin{aligned} y &= \frac{8x+5}{2x+3} \\ 2xy + 3y &= 8x + 5 \\ 2xy - 8x &= 5 - 3y \\ x &= \frac{5-3y}{2y-8} \\ \mathrm{f}^{-1}\left(\frac{3}{2}\right) &= \frac{5 - 3\left(\frac{3}{2}\right)}{2\left(\frac{3}{2}\right) - 8} = -\frac{1}{10} \end{aligned}$$

(2)

b. Show that

$$\mathrm{f}(x) = A + \frac{B}{2x+3}$$

where *A* and *B* are constants to be found.

$$\begin{aligned} \mathrm{f}(x) &= \frac{8x+5}{2x+3} \\ &= \frac{4(2x+3) - 7}{2x+3} \\ &= 4 - \frac{7}{2x+3} \\ A &= 4,\ B = -7 \end{aligned}$$

(2)

The function g is defined by

$$\mathrm{g}(x) = 16 - x^2 \qquad 0 \leqslant x \leqslant 4$$

c. State the range of g^{-1}

$$0 \leqslant y \leqslant 4$$

(1)

d. Find the range of fg^{-1}

$$\begin{aligned} \mathrm{f}(0) &= \frac{5}{3} \\ \mathrm{f}(4) &= \frac{8(4)+5}{2(4)+3} = \frac{37}{11} \end{aligned}$$

$$\text{range of } \mathrm{fg}^{-1} \quad \frac{5}{3} \leqslant y \leqslant \frac{37}{11}$$

(3)

11.

Prove, using algebra, that

$$n\left(n^2+5\right)$$

is even for all $n \in \mathbb{N}$

Assume *n* is odd,

$$\begin{aligned} n &= 2k+1,\ k \in \mathbb{Z}^+ \\ n\left(n^2+5\right) &= (2k+1)(2k+1)^2+5 \\ &= (2k+1)\left(4k^2+4k+6\right) \\ &= 2(2k+1)\left(2k^2+2k+3\right) \\ &= \text{even} \end{aligned}$$

Assume *n* is even,

$$\begin{aligned} n &= 2k,\ \ k \in \mathbb{Z}^+ \\ n\left(n^2+5\right) &= (2k)\left[(2k)^2+5\right] \\ &= 2\left[4k^3+5k\right] \\ &= \text{even} \end{aligned}$$

Therefore $n\left(n^2+5\right)$ is even for all $n \in \mathbb{N}$

(4)

12.

The function f is defined by

$$\mathrm{f}(x) = \frac{\mathrm{e}^{3x}}{4x^2+k}$$

where *k* is a positive constant.

a. Show that

$$\mathrm{f}'(x) = \left(12x^2-8x+3k\right)\mathrm{g}(x)$$

where $\mathrm{g}(x)$ is a function to be found.

$$\begin{aligned} \mathrm{f}'(x) &= \frac{3\mathrm{e}^{3x}\left(4x^2+k\right)-\mathrm{e}^{3x}(8x)}{\left(4x^2+k\right)^2} \\ &= \frac{\mathrm{e}^{3x}\left(12x^2+3k-8x\right)}{\left(4x^2+k\right)^2} \\ \mathrm{g}(x) &= \frac{\mathrm{e}^{3x}}{\left(4x^2+k\right)^2} \end{aligned}$$

(3)

Given that the curve with equation $y=\mathrm{f}(x)$ has at least one stationary point,

b. Find the range of possible values of *k*.

$$\begin{aligned} \mathrm{f}'(x) &= 0 \\ \mathrm{g}(x) &\neq 0 \\ 12x^2-8x+3k &= 0 \\ (-8)^2-4(12)(3k) &\geqslant 0 \\ k &\leqslant \frac{4}{9} \end{aligned}$$

(3)

13.

Relative to a fixed origin *O*

- the point *A* has position vector $4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$
- the point *B* has position vector $4\mathbf{j} + 6\mathbf{k}$
- the point *C* has position vector $-16\mathbf{i} + p\mathbf{j} + 10\mathbf{k}$

where *p* is a constant.

Given that *A*, *B* and *C* lie on a straight line,

a. find the value of *p*.

$$AB = B - A = \begin{pmatrix}0\\4\\6\end{pmatrix} - \begin{pmatrix}4\\-3\\5\end{pmatrix} = \begin{pmatrix}-4\\7\\1\end{pmatrix}$$

$$AC = C - A = \begin{pmatrix}-16\\p\\10\end{pmatrix} - \begin{pmatrix}4\\-3\\5\end{pmatrix} = \begin{pmatrix}-20\\p+3\\5\end{pmatrix}$$

$$AB = kAC$$

$$\begin{pmatrix}-4\\7\\1\end{pmatrix} = k\begin{pmatrix}-20\\p+3\\5\end{pmatrix}$$

$$k = \frac{1}{5}$$

$$p = 32$$

(3)

The line segment *OB* is extended to a point *D* so that $\overrightarrow{CD}$ is parallel to $\overrightarrow{OA}$

b. Find $\left|\overrightarrow{OD}\right|$, writing your answer as a fully simplified surd.

$$\lambda OB = OD$$

$$D = \lambda\begin{pmatrix}0\\4\\6\end{pmatrix}$$

$$CD = \mu OA$$

$$\lambda\begin{pmatrix}0\\4\\6\end{pmatrix} - \begin{pmatrix}-16\\32\\10\end{pmatrix} = \mu\begin{pmatrix}4\\-3\\5\end{pmatrix}$$

$$\mu = 4$$

$\mathbf{j}$ component

$$4\lambda - 32 = -12$$

$$\lambda = 5$$

$\mathbf{k}$ component

$$\text{LHS} = 5(6) - 10 = 20$$

$$\text{RHS} = 4(5) = 20$$

$$\left|\overrightarrow{OD}\right| = \sqrt{16^2 + 12^2 + 20^2} = 20\sqrt{2}$$

(3)

14.

a.

Express $\frac{3}{(2x-1)(x+1)}$ in partial fractions.

$$\begin{aligned}\frac{3}{(2x-1)(x+1)} &= \frac{A}{2x-1}+\frac{B}{x+1}\\ 3 &= A(x+1)+B(2x-1)\\ x &= -1\\ 3 &= B(-3) \quad \implies B=-1\\ x &= \frac{1}{2}\\ 3 &= A\left(\frac{3}{2}\right) \quad \implies A=2\\ \frac{3}{(2x-1)(x+1)} &= \frac{2}{2x-1}-\frac{1}{x+1}\end{aligned}$$

(3)

When chemical A and chemical B are mixed, oxygen is produced.

A scientist mixed these two chemicals and measured the total volume of oxygen produced over a period of time.

The total volume of oxygen produced, $V\,\mathrm{m}^3$, t hours after the chemical were mixed, is modelled by the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{3V}{(2t-1)(t+1)} \qquad V \geqslant 0 \qquad t \geqslant k$$

where k is a constant.

Given that exactly 2 hours after the chemicals were mixed, a total volume of $3\,\mathrm{m}^3$ of oxygen had been produced,

b. Solve the differential equation to show that

$$V=\frac{3(2t-1)}{(t+1)}$$

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{3V}{(2t-1)(t+1)}\\ \int\frac{1}{V}\mathrm{d}V &= \int\frac{3}{(2t-1)(t+1)}\mathrm{d}t\\ \ln V &= \ln(2t-1)-\ln(t+1)+c\\ t &= 2,\ V=3\\ \ln 3 &= \ln 3-\ln 3+c\\ \ln V &= \ln(2t-1)-\ln(t+1)+\ln 3\\ V &= \frac{3(2t-1)}{t+1}\end{aligned}$$

(5)

The scientist noticed that

- there was a **time delay** between the chemicals being mixed and oxygen being produced
- there was a **limit** to the total volume of oxygen produced

Deduce from the model

c.

i. the **time delay** giving your answer in minutes,

$$\begin{aligned} \text{when } V &= 0, \\ 3(2t-1) &= 0 \\ t &= 0.5 \text{ hours} \\ \text{time delay} &= 30 \text{ minutes} \end{aligned}$$

ii. the **limit** giving your answer in m^3

$$V = \frac{6t-3}{t+1} = \frac{6(t+1)-9}{t+1} = 6 - \frac{9}{t+1}$$

$$\text{limit} = 6\,\mathrm{m}^3$$

(2)

15.

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Given that the first three terms of a geometric series are

$12\cos\theta$ $\quad 5+2\sin\theta \quad$ and $\quad 6\tan\theta$

a. Show that

$$4\sin^2\theta - 52\sin\theta + 25 = 0$$

$$\begin{aligned} \frac{5+2\sin\theta}{12\cos\theta} &= \frac{6\tan\theta}{5+2\sin\theta} \\ 25 + 20\sin\theta + 4\sin^2\theta &= \frac{6\sin\theta}{\cos\theta}(12\cos\theta) \\ 4\sin^2\theta - 52\sin\theta + 25 &= 0 \end{aligned}$$

(3)

Given that θ is an obtuse angle measured in radians,

b. Solve the equation in part (a) to find the exact value of θ

$$\begin{aligned} (2\sin\theta - 25)(2\sin\theta - 1) &= 0 \\ \sin\theta &\neq \frac{25}{2} \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{5\pi}{6} \end{aligned}$$

(2)

c. Show that the sum to infinity of the series can be expressed in the form

$$k\left(1-\sqrt{3}\right)$$

where *k* is a constant to be found.

$$a = 12\cos\frac{5\pi}{6} = -6\sqrt{3}$$

$$r = \frac{5+2\sin\frac{5\pi}{6}}{-6\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$S_\infty = \frac{-6\sqrt{3}}{1-\left(-\frac{1}{\sqrt{3}}\right)}$$

$$= \frac{-18}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{18\left(1-\sqrt{3}\right)}{3-1}$$

$$k = 9$$

(5)

16.

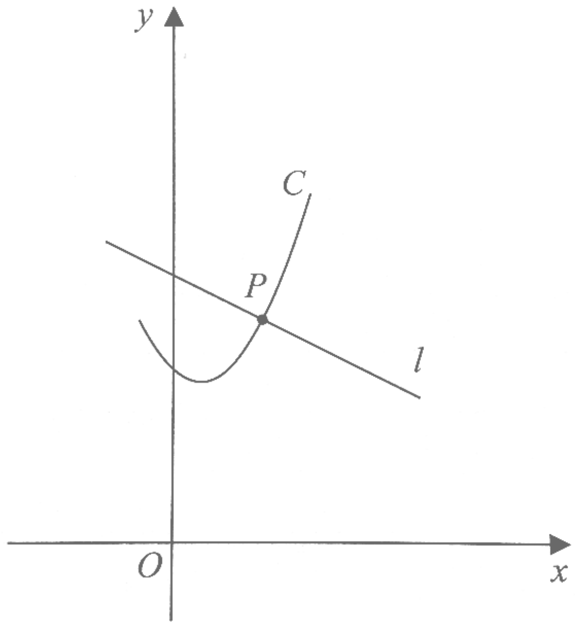

Figure 6

Figure 6 shows a sketch of the curve *C* with parametric equations

$$x = 2\tan t + 1 \qquad y = 2\sec^2 t + 3 \qquad -\frac{\pi}{4} \leqslant t \leqslant \frac{\pi}{3}$$

The line *l* is the normal to *C* at point *P* where $t = \frac{\pi}{4}$

a. Using parametric differentiation, show that an equation for *l* is

$$y = -\frac{1}{2}x + \frac{17}{2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4\sec t(\sec t\tan t)}{2\sec^2 t} = 2\tan t$$

$$\text{when } t = \frac{\pi}{4}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2,\ \text{gradient of nomral} = -\frac{1}{2}$$

$$y = \frac{2}{\left(\cos\frac{\pi}{4}\right)^2} + 3 = 7,\ x = 2\tan\frac{\pi}{4} + 1 = 3$$

$$y - 7 = -\frac{1}{2}(x - 3)$$

$$y = -\frac{1}{2}x + \frac{17}{2}$$

(5)

b. Show that all points on *C* satisfy the equation

$$y = \frac{1}{2}(x-1)^2 + 5$$

$$\begin{aligned} \sin^2 t + \cos^2 t &= 1 \\ \tan^2 t + 1 &= \sec^2 t \\ y &= 2\left(\tan^2 t + 1\right) + 3 \\ &= 2\left[\left(\frac{x-1}{2}\right)^2 + 1\right] + 3 \\ &= 2\left(\frac{1}{4}(x-1)^2 + 1\right) + 3 \\ &= \frac{1}{2}(x-1)^2 + 5 \end{aligned}$$

(2)

The straight line with equation

$$y = -\frac{1}{2}x + k$$ where *k* is a constant

Intersects C at two distinct points.

c. Find the range of possible values for *k*.

$$\begin{aligned} \frac{1}{2}(x-1)^2 + 5 &= -\frac{1}{2}x + k \\ x^2 - 2x + 1 + 10 &= -x + 2k \\ x^2 - x + 11 - 2k &= 0 \\ (-1)^2 - 4(1)(11-2k) &> 0 \\ 1 - 44 + 8k &> 0 \\ k &> \frac{43}{8} \end{aligned}$$

(5)