Practice Problems

1. In this geometric sequence, each term is twice as large as the one before it. This means that the 5th term will be twice as large as the 4th term, which is 16. Therefore, the 5th term is 16*2 = **32**.
2. Expand $(1.6 + 5)^2$ to get $1.6^2 + 2(5)(1.6) + 5^2$. After subtracting the 1.6^2 and 5^2, we are left with 2(5)(1.6) = **16**.
3. We can think of this like a geometric series in that 0.712121212... can be thought of as 0.7 + 0.012 + 0.0012 + ... As such, we might think of using a similar strategy to deal with repeating decimals. Let S be our repeating decimal. We have that 100S = 71.2121212..., just shifting the decimal point right two places. Then subtract the original 0.7121212 from this to get 70.5 since a lot of the "12" terms line up and cancel out. This value is 99 times larger than the original decimal because we multiplied it 100 times and subtracted it once. Therefore, the final answer is 70.5/99 which simplifies to **47/66**.
4. Because its absolute value is 2, x^2-12x+34 may equal 2 or -2, so we solve both cases. For 2, we get the equation x^2-12x+34=2, which becomes x^2-12x+32=0 —> (x-8)(x-4)=0. This means that 4 and 8 are two solutions. Now we can solve for x^2-12x+32=-2. This becomes x^2-12x+36=0 —> $(x-6)^2$=0. This gives us 6 as a solution. The sum of all possible solutions is 4 + 6 + 8 = **18**.
5. 207^2 may be written as the square of a binomial like $(200+7)^2$. Expanding, we get 200^2 + 2(200)(7) + 49. As requested in the problem, we have 200 multiplied by something and adding 49. Now we just factor out a 200 from 200^2+2(200)(7) to get 200(200+14). Therefore, **14** is our answer.
6. Since even numbers is an arithmetic series, we may use our trick to think about the sum of 25 consecutive even integers as 25 times the average of the list. This median can be computed by dividing 10000 by 25 to get 400. We know that since there are 25 numbers total, there are 12 numbers larger than this middle value, and that each one is 2 greater than the previous. This means the largest value is 2(12) greater than 400, which is **424**.
7. By the rule, the terms of the sequence starting from a_1 are 10^2, 10^4, 10^8, etc. As such, multiplying together a_1 through a_n gives us $10^{2+4+8+16+...} = 10^{(2^\wedge(n+1))-2}$ by simplifying the geometric series. Since it's a power of 10, the number of digits this has is 2^{n+1}-2 + 1, which we want to be 100 or above. Thus, we want $2^{n+1} \geq 101$. 2^6 is 64 and 2^7 is 128, so 7 is the minimal value for n+1. Since n+1 ≥ 7, n≥6 so the minimal value that works for n is **6**.

8. Solution 1 (combining squaring and cubing):

We are told that $t+\frac{1}{t}=3$. So, it follows that $\left(t+\frac{1}{t}\right)^2=3^2 \rightarrow t^2+2+\frac{1}{t^2}=9 \rightarrow t^2+\frac{1}{t^2}=7$. So, $\left(t+\frac{1}{t}\right)^3=\left(t^2+\frac{1}{t^2}\right)\left(t+\frac{1}{t}\right)=7(3) \rightarrow t^3+t+\frac{1}{t}+\frac{1}{t^3}=21 \rightarrow t^3+3+\frac{1}{t^3}=21 \rightarrow t^3+\frac{1}{t^3}=18$. Finally, $\left(t^2+\frac{1}{t^2}\right)\left(t^3+\frac{1}{t^3}\right)=7(18) \rightarrow t^5+t+\frac{1}{t}+\frac{1}{t^5}=126 \rightarrow t^5+3+\frac{1}{t^5}=126 \rightarrow t^5+\frac{1}{t^5}=\mathbf{123}$.

Solution 2 (fifth power directly):

Alternatively, using the binomial theorem yields $27=3^3=\left(t+\frac{1}{t}\right)^3=t^3+3\left(t+\frac{1}{t}\right)+\frac{1}{t^3}=t^3+9+\frac{1}{t^3}$, so $t^3+\frac{1}{t^3}=27-9=18$. Also, $243=3^5=\left(t+\frac{1}{t}\right)^5=t^5+5t^3+10\left(t+\frac{1}{t}\right)+5\frac{1}{t^3}+\frac{1}{t^5}$. Thus, $t^5+\frac{1}{t^5}=243-5\left(t^3+\frac{1}{t^3}\right)-10\left(t+\frac{1}{t}\right)=243-5\times 18-10\times 3=\mathbf{123}$.

9. Let a and b be the cube roots of x and y respectively. We have that a-b = 2 and $a^3-b^3=(a-b)(a^2+ab+b^2)=12$. Substituting 2 for a-b we have $2(a^2+ab+b^2)=12$ so $a^2+ab+b^2=6$. Since we know a-b, we may square it to obtain $(a-b)^2=a^2-2ab+b^2$, which is $2^2=4$. Subtracting this from $a^2+ab+b^2=6$, we get that 3ab=2, so ab = ⅔. Undoing our original substitution, we have $xy=a^3b^3=(⅔)^3=$ **8/27**.
10. Let y be what we want to compute. Observe that if we multiply y by 3 we have a difference of squares which simplifies to 3y = product of the radicals = $(49-x^2)-(25-x^2)$ = 49-25 = 24. Since 3y=24, we can solve to get **y=8**.

Challenge Problems

1. Solution 1 (legit): Notice that if we multiply the whole thing by 3-2 = 1, by difference of squares the first factor becomes 3^2-2^2, which conveniently combines by difference of squares with the second factor to get 3^4-2^4, which combines with the third factor, and so on, so we end up with C) $\mathbf{3^{128}-2^{128}}$.
“Solution” 2 (answer choices): Multiply it out for (2+3), $(2+3)(2^2+3^2)$, etc to try to see a pattern, since each time it looks like a 3^n-2^n, so it makes sense to guess C. This guessing tactic is colloquially known as engineer’s induction.
2. Solution 1: Notice that Andy moves east every four moves, north every four moves, west every four moves, and so on. Since we can add things in any order we want, we can just organize this into four separate arithmetic series. The distance Andy moves east is 1+5+9+...+2017, north is 2+6+10+...+2018, west is 3+7+11+2019, and south is 4+8+...+2020. Since north/south and east/west cancel each other, we subtract them to get that Andy moves -1010 in the x-direction and -1010 in the y-direction, so we just combine these with his initial coordinates to get that he ends up at **(-1030, -990)**.
Solution 2: Every four moves, both of Andy’s coordinates decreases by 2, and since this happens 505 times both his coordinates decrease by 1010 so he ends up at (-1030, -990).

3. Let Henry end up oscillating between points P and Q, with P being closer to home. These distances are A and B respectively as given by the problem. By symmetry, the distance from P to home is equal to the distance from Q to the gym. As such, A = 2-B. When Henry walks toward home from point Q, he walks ¾ of the distance to end up at point P where he turns around, so we have the equation B-A=¾B. Solving this system, we get B=8/5 and A=⅖ so |A-B| = **6/5.**
4. Solution: Noticing the final condition of the problem that f(p) = p for prime numbers, we think about prime factorization. Option E stands out in that it is the only choice whose numerator is not a prime number, so we examine this option. Since 25 can be prime factored as 5*5, we apply the properties from the problem statement to get that
f(25) = f(5)+f(5) = 5+5 = 10. With the first condition in mind, we may also write f(25) as f(25/11 * 11) = f(25/11) + f(11) = f(25/11) + 11 since 11 is prime. Combining these results, we have that f(25/11) + 11 = 10 which we can solve to get that f(25/11) is -1. Since this is negative, option **E** is correct and we are done.
5. Solution 1 (split into components):

We plot this on the coordinate grid with point O as the origin. We will keep a tally of the x-coordinate and y-coordinate separately.

First move: The ant moves right 5. Second move: We use properties of a $30-60-90$ triangle to get $\frac{5}{4}$ right, $\frac{5\sqrt{3}}{4}$ up. Third move: $\frac{5}{8}$ left, $\frac{5\sqrt{3}}{8}$ up. Fourth move: $\frac{5}{8}$ left. Fifth move: $\frac{5}{32}$ left, $\frac{5\sqrt{3}}{32}$ down. Sixth move: $\frac{5}{64}$ right, $\frac{5\sqrt{3}}{64}$ down.

Total of x-coordinate: $5+\frac{5}{4}-\frac{5}{8}-\frac{5}{8}-\frac{5}{32}+\frac{5}{64}=\frac{315}{64}$. Total of y-coordinate: $0+\frac{5\sqrt{3}}{4}+\frac{5\sqrt{3}}{8}+0-\frac{5\sqrt{3}}{32}-\frac{5\sqrt{3}}{64}=\frac{105\sqrt{3}}{64}$.

After this cycle of six moves, all moves repeat with a factor of $(\frac{1}{2})^6=\frac{1}{64}$. Using the formula for a geometric series, multiplying each sequence by $\frac{1}{1-\frac{1}{64}}=\frac{64}{63}$ will give us the point P.

$\frac{315}{64}\cdot\frac{64}{63}=5$, $\frac{105\sqrt{3}}{64}\cdot\frac{64}{63}=\frac{5\sqrt{3}}{3}$. Therefore, the coordinates of point P are $(5,\frac{5\sqrt{3}}{3})$, so using the Pythagorean Theorem, $OP^2=\frac{100}{3}$, for an answer of $\boxed{103}$.

Solution 2 (complex rotation): On the complex plane, we rotate 60 degrees counterclockwise by multiplying by $e^{i\pi/3}$, so for the ant's movement we have the series $5 + 5e^{i\pi/3}/2 + 5(e^{i\pi/3}/2)^2 + \ldots$, which is an infinite geometric series equal to $\frac{5}{1-\frac{1}{2}e^{\frac{i\pi}{3}}}=\frac{5}{1-\frac{1+i\sqrt{3}}{4}}=\frac{20}{3-i\sqrt{3}}=5+\frac{5i\sqrt{3}}{3}$ What we want is the magnitude of this, squared which is 25 + 25/3 = 100/3, so the answer is 100 + 3 = **103**.